NOTES ON ABELIAN CHERN SIMONS THEORY (FOR HOMOTOPY THEORISTS)

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1. Introduction

The mathematical facts worthy of being studied are those which, by their analogy with other facts, are capable of leading us to the knowledge of a physical law. They reveal the kinship between other facts, long known, but wrongly believed to be strangers to one another.

—Henri Poincare

These notes explore the relationship between linking numbers and certain observables in a topological quantum field theory (TQFT). This relationship can be most clearly seen in one of the simplest gauge theory, commonly referred to as "Abelian Chern-Simons Theory." Towards this end, our strategy will be to construct a mathematical model of this theory, via the Batalin-Vilkovisky formalism, following [CG], [1]. using a combination of differential geometry and homotopical algebra.

These notes are taking from a lecture of Jeremy Mann, by Jerome O'Brien. Any mistake is due to the note-taker.

1.1. **A Note on Technical Difficulties.** Constructions of such models are notoriously subtle, and our model will contain a myriad of physically meaningless constituents. These redundancies fall into two types.

Every physical theory has a (often implicit) scale at which it is valid. For example, Chern-Simons theory is only valid at low energy/large length [2] scales. One could therefore argue that the degrees of freedom of an ideal model of this theory would not include rapidly varying degrees of freedom. Our model, on the other hand, will make reference to such degrees of freedom; it will be a "continuum" model. [3] These superfluous degrees of freedom will permit us smooth, flabby methods, at the cost of technical difficulties the author encourages the reader to ignore.

Every physical theory comes with a (often implicit) phenomenological interpretation. In particular, a notion of which quantities can be experimentally verified. One may argue there are no differences between degrees of freedom which couple to experimental apparatuses in an identical fashion, thereby leading to the same outcome in any possible experiment. Following this logic, an ideal model should not contain such "redundant" degrees of freedom. Our model, on the other hand, will make explicit reference to such redundant degrees of freedom.

Homotopical methods will remedy these deficits, through a standard technique of linguistic subversion and reappropriation, all of which will be implicit in our discussion.

1.2. **Outline.** As this will be a mathematical approach, we begin with general considerations, specializing as we go along. We warn the reader that our discussion will be very back-of-the-envelope. Rigorous details may be found in [C] and [CG]. Everything should be interpreted in a "derived" sense.

We begin our discussion by reviewing facets of Poincare duality, in a language convenient for our purposes. In order to be compatible with smooth methods, our incarnation of Poincare Duality will be at the level of differential forms. We invite the reader to consult [BT] for a detailed introduction to these methods.

We then "shift" these ingredients, recasting them in a light more amenable to symplectic geometry. As in the classical case, these symplectic structures will form the basic ingredients of our "classical theory". Here, classical means the absence of correlations between observable quantities (in the sense of probability theory). In other words, a "mean field theory approximation" of what we are really interested in.

Next, we parallel Dirac's quantization proceduce, producing a model of our object of interest. In order to interpret our results, we define the notion of an expectation value and correlation in this setting. After reviewing various equivalent notions of the linking number, we outline the promised relationship between between observables in a TQFT and linking numbers.

We end with a discussion of how this approach relates to Feynman and Schwinger's approach to quantization, along with a whisper relating these phenonomena to a more contemporary approach: factorization homology.

2. Recollections of Poincare Duality

> *Every theoretical physicist that's any good knows six or seven different theoretical representations for exactly the same physics, and knows that they are all equivalent, and that nobody is ever going to be able to decide which one is right at that level, but keeps them in [their] head, hoping that they will give him different ideas for guesses.*
>
> —Richard Feynman

We model our primary antagonist as the cosheaf of compactly supported differential forms, with it's natural deRham differential: [4]

$$(\Omega_c^\bullet, \mathrm{d}_{\mathrm{dR}})$$

An orientation gives differential forms [5] a natural pairing, which may be modeled via "wedge and integrate":

$$\int(- \wedge -) : \mathrm{Sym}^2(\Omega_c^\bullet) \xrightarrow{(-\wedge -)} \Omega_c^\bullet \xrightarrow{\int} \mathbb{R}[-n]$$
$$\eta \cdot \gamma \longmapsto \int \eta \wedge \gamma \in \mathbb{R}[-n]$$

[6]

Furthermore, an analytic theorem of deRham states that the natural adjoint map:

$$\Omega_c^\bullet \xrightarrow{\simeq} (\Omega_c^\bullet)^\vee[-n]$$
$$\eta \longmapsto \int \eta \wedge (-)$$

is an equivalence. As these maps are natural with respect to extension by zero and the second symmetric power of the deRham differential, we can view these as maps of cosheaves. [7] In other words, Poincare duality births a symmetric, nondegenerate pairing. [8]

Example 1. For example, in $\mathbb{R}^3$ with it's standard orientation, dx is dual to $dy \wedge dz$. This follows from the standard "right hand rule".

In summary, we may view this recipe as defining a locally varying nondegenerate symmetric pairing of degree -n:

$$\int_{(-)}(- \wedge -) \in \mathrm{Sym}^2\big((\Omega_c^\bullet)^\vee\big)[-n]$$
$$\simeq \Big(\mathrm{Sym}^2\big((\Omega_c^\bullet[1]^\vee)[-1]\big)\Big)[-n]$$

3. A Symplectic Articulation of Poincare Duality

...physicist care about field theory because it is a way of talking about systems with many degrees of freedom, labeled by points in a geometric space, and interacting in a way that respects the notions of homogeneity and locality associated to the geometry of that space

—Ingmar Saberi, [S]

Our next goal will be to construct a linear symplectic object from the above data, along with it's associated commutative Poisson algebra.

3.1. The Symplectic Form.

We begin by recasting the above symmetric pairing as skew-symmetric. This is accomplished through a purely formal maneuver: the "decalage isomorphism" used in Koszul Duality. In general, this provides a linear equivalence:

$$\mathrm{Sym}^k(V[-1]) \simeq \Lambda^k(V)[-k]$$

In other words, a shifted antisymmetric tensor is a symmetric tensor on a shift.

In our case ($k = 2, V = \Omega_c^\bullet[1]$), this decalage translates our symmetric pairing of degree $-n$ as a skew symmetric pairing of degree $2 - n$ on a shift:

$$\Lambda^2(\Omega_c^\bullet[1]) \longrightarrow \mathbb{R}[2 - n]$$

so that 1-forms are now in degree 0. [9] Moreover, the nondegeneracy of the original symmetric pairing implies that this skew symmetric form is in fact symplectic.

In summary, we have produced a cosheaf of symplectic vector spaces [10] [11] on oriented n-manifolds of any (but fixed) dimension:

Definition 2 (Fields)**.** We define the *fields of Chern-Simons theory* as the cosheaf of symplectic vector spaces:

$$\mathcal{E}_{\mathrm{CS}} = \Big(\Omega_c^\bullet[1], \int (- \wedge -) \in \Lambda^2\big((\Omega_c^\bullet[1])^\vee\big)[2 - n]\Big)$$

Where the symplectic form has degree $2-n$, as demanded by our combination of Poincare and Koszul Duality. Note that when $n = 2$, this symplectic symplectic form is degree 0, and coincidesw with the Atiyah-Bott symplectic form on the moduli space of flat connections on a surface. When $n = 3$, this the form is in degree -1.

This object will constitute a model of the "phase space" of our "fields". The term "fields" should be taken to mean models for degrees of freedom of some system we'd like to observe/measure. [12] Therefore, our next goal will be to model all the possible observable quantities of our system. Whatever an observable is, it should eat fields and spit out numbers, in a manner compatible with the relevant symmetries. [13]

Moreover, we'd like our model to encode the manner in which two degrees of freedom are physically indistinguishable. Dually, our model must be able to recognize the manner in which two observations will always produce the same number. To accomplish our primary goal, we begin by constructing an approximation for our observables of interest, valid in a "classical" regime. We'll refer to this as

4. The Commutative Algebra of Observables

My (algebraic) methods are really methods of working and thinking; this is why they have crept in everywhere anonymously

— Emmy Noether

In our setting this is straightforward, as functions on a vector space are adequately approximated by polynomial functions. With this in mind,

Definition 3 (Classical Observables)**.** We define the *classical observables of Chern-Simons theory* as the copresheaf of commutative algebras: [14]

$$\mathrm{Obs}^{\mathrm{cl}}_{\mathrm{CS}} = \mathrm{CE}^*(\mathrm{T}_0[-1]\mathcal{E}_{\mathrm{CS}}) = \mathrm{Sym}^\bullet\big((\Omega^\bullet_c[1])^\vee\big)^{15}$$

In order to get a better feel for these types of functions, let's construct a related family of examples. For example, $n+1$ points of $\mathbb{R}^k$, $x : [k] \to \mathbb{R}^n$, gives a degree $k+1$ polynomial on sections of the of the trivial $\mathbb{R}$-line bundle, given by:

$$\phi \mapsto \phi(x_0)\dots\phi(x_k)$$

More generally, given a vector space V, and k linear functionals $v : [k] \to V^\vee$, the symmetrization of

$$\phi \mapsto v^0(\phi(x_0))\dots v^n(\phi(x_k))$$

gives a degree $k+1$ polynomial on sections of the of the trivial V bundle. [16] An analogous line of thought leads to, for any vector bundle E over an oriented manifold M, a dense inclusion:

$$\Gamma(M^{\times k}, (E^\vee)^{\boxtimes k})_{\Sigma_k} \hookrightarrow \mathrm{Sym}^k(\mathcal{E}^\vee)$$

which we leave for the reader to spell out. Additional examples of these functions will be spelled out in coming sections.

Now that we have a model of our observable quantities, we'd like to understand how measurements of our system change in response to interactions with external systems. This is accomplished by describing the induced

4.1. **Poisson Bracket On Observables.** Our Poisson bracket is a biderivation, which is induced by the dual of the symplectic form.

If you know the behavior of a biderivation on linear functions, you know the biderivation. So, in order to get a concrete grasp of our Poisson bracket, we will need to compute a map of the form:

$$\Lambda^2\Big((\Omega^\bullet_c[1])^\vee\Big) \xrightarrow{\{\cdot,\cdot\}} \mathbb{R}[2-n]$$

Which we do by specifying the equivalent data:

$$\mathrm{Sym}^2\Big((\Omega^\bullet_c)^\vee\Big) \xrightarrow{\{\cdot,\cdot\}} \mathbb{R}[n]$$

As the essential ingredients in deriving a description of the above map is densely populated with technical difficulties, and spelled out in a variety of sources, [CG2] we'll simply give two descriptions of the answer(s).

Specifying the behavior on all pairs of observables in our model is cumbersome, and irrelevant for the present discussion. However, there are two equivalent subclasses of linear observables in our model which admit a straightforward description.

The first is modelled by realizing the dual space as a suitable shift of compactly supported forms:

$$\Omega^\bullet_c[n-1] \overset{\simeq}{\hookrightarrow} \big((\Omega^\bullet_c)[1]\big)^\vee$$
$$\eta \longmapsto \Big(\gamma \mapsto \int \eta \wedge \gamma\Big)$$

In this model, the pairing is equivalent to the "same" wedge and integrate formula:

$$\mathrm{Sym}^2\Big(\Omega^\bullet_c[n-1]\Big)\Big) \to \mathbb{R}[n-2]$$
$$\eta_0 \cdot \eta_1 \mapsto \int \eta_0 \wedge \eta_1$$

The second description is given by transverse intersection of suitably closed oriented submanifolds. [17] Here, we're viewing a submanifold as a linear observable the obvious way:

$$\eta \mapsto \int_N \eta \in \mathbb{R}[\dim(N) - 1]$$

so that loops are degree 0 elements of the dual.

Moreover, Stokes' Theorem tells us that the boundary of the submanifolds, computes the dual of the deRham differential. In particular, null-cobordisms provide nullhomotopies of the above family of observables.

One goes from an oriented submanifolds to compactly supported forms by proving the existence of a suitable Thom class of its normal bundle.

5. Quantization

There seeemed to be a close relation between the Poissionn bracket of quantities and the commutator. The idea came in a flash, I suppose, and provided of course some excitement, and then came the reaction: "No, this is probably wrong".

—P.M. Dirac

Our next step will be to quantize the above system, via Dirac's recipe for quantization. More precisely, this quantization ansatz states that we should interpret the *Poisson bracket on the dual of a symplectic vector space as a central extension of Lie algebras.*

We can now follow the standard procedure of revising our interpretation, seeing it as classifying a central extension of the dual of our fields (of degree $2 - n$): the Heisenberg Lie algebra, conventionally denoted:

$$(\mathbb{R} \cdot \hbar)[2 - n] \to \mathrm{Heis}(\mathcal{E}_{\mathrm{CS}}) \to \left(\Omega_c^\bullet[1]\right)^\vee$$

The underlying vector space of this lie algebra may be split into:

$$\mathrm{Heis}(\mathcal{E}_{\mathrm{CS}}) \simeq \left(\Omega_c^\bullet[1]\right)^\vee \oplus (\mathbb{R} \cdot \hbar)[2 - n]$$

[18] Now that we have our Lie algebra, we can use the Chevalley-Eilenberg construction to exponentiate our Lie algebra into a cococommutative coalgebra. [19]

Definition 4. We define the (perturbative) *quantum observables of abelian Chern-Simons theory* to be: [20]

$$\begin{aligned}
\mathrm{Obs}^{\mathrm{q}}_{\mathrm{CS}} &= \mathrm{CE}_*\Big(\mathrm{Heis}(\mathcal{E}_{\mathrm{CS}})\Big) \\
&\simeq \Big(\mathrm{Sym}(\mathcal{E}_{\mathrm{CS}}[1]), \mathrm{d}_{\mathrm{CE}} =: \mathrm{d}^{q}_{\mathrm{CS}}\Big) \\
&\simeq \Big(\mathrm{Sym}(\Omega_c^\bullet[n-1] \oplus \hbar[3-n]), d_{\mathrm{dR}} + \hbar \int (- \wedge -) =: \mathrm{d}_{\mathrm{dR}} + \hbar\Delta_{\mathrm{BV}})
\end{aligned}$$

Here, the Poincare pairing is extended as a coderivation of this coalgebra, which behaves as a (formal) second-order differential operator. This is conventionally referred to as the *BV Laplacian* when n = 3. We now restrict to $n = 3$ for the rest of our discussion.

As our observables come with a weight filtration [21], we obtain a spectral sequence. Note that the d_2 differential is given by the BV Laplacian a.k.a the Poincare pairing.

5.1. **Expectation Values.** When $n = 3$, $\hbar$, (formal) polynomials of 2-forms in the first model of the dual space) and loops (in the second model of the dual space) are in degree zero. We take this to be the case for the rest of our discussion.

In this case, the functoriality of the Chevalley-Eilenberg constrution gives a fibre sequence of coalgebras:

$$\mathbb{R}[\hbar] \to \mathrm{Obs}^{\mathrm{q}}_{\mathrm{CS}} \xrightarrow{\hbar=0} \mathrm{Sym}^{\bullet}(\mathcal{E}^{\vee}_{\mathrm{CS}})$$

This map allows us to define the [22] expectation value of an observable:

Definition 5. Given an observable $\mathcal{O} \in \mathrm{Obs}^{\mathrm{q}}_{\mathrm{CS}}$, we define it's *expectation value*, $< \mathcal{O} >$, [23] to be a solution to the lifting problem:

$$\begin{tikzcd} & \mathbb{R}[\hbar] \arrow[d] \\ \mathbb{R} \arrow[ur, "<\mathcal{O}>"] \arrow[r, "\mathcal{O}"] & \mathrm{Obs}^{q}_{\mathrm{CS}} \end{tikzcd}$$

Given two observables with disjoint support, $\mathcal{O}_1, \mathcal{O}_2$, we define their *correlation* to be:

$$< \mathcal{O}_1\mathcal{O}_2 >_c = < (\mathcal{O}_1 - < \mathcal{O}_1 >)(\mathcal{O}_2 - < \mathcal{O}_2 >) >$$

Here, we are using the "multiplication" in the symmetric "algebra".

In our model, this is saying that computing the expectation value of an observable is equivalent to solving the (under-determined) equation:

$$\mathrm{d}^q(?) = \mathcal{O} - (\sum_{i \geq 0} (?_i)\hbar^i)$$

Where each $?_i$ is a real number.

Note that the expectation value of a coboundary is necessarily zero, so that *homotopic observables have the same expectation values.* Therefore, the homotopical character of our objects encode how observations are physically indistinguishable.

6. Linking Numbers

The manner in which a closed form is zero in cohomology contains geometric information

—Shiing-Shen Chern, [DGMS]

Now that we have the formal stuff out of the way, we now describe Abelian Chern-Simons theory's famed relationship to linking numbers . Our first step will be to construct some observables living in $\mathbb{R}^3$. Whatever these are, they should eat fields and spit out numbers:

$$\mathcal{O} : \Omega^{\bullet}_c[-1](\mathbb{R}^3) \to \mathbb{R}$$

A general class of these examples come from embedded oriented links $L : \mathbb{S}^1 \to \mathbb{R}^3$:

$$A \longmapsto \mathcal{O}_L(A) = \int_{\mathrm{S}^1} A|_L$$

which we can imagine arising as the *difference* between two curves that start and end at the same place.

The Fundamental Theorem of Calculus tells us that, as S^1 has no boundary, it in fact defines a map:

$$\mathcal{O}_L : \mathbb{R}[0] \to \mathrm{Obs}^{\mathrm{q}}_{\mathrm{CS}}(\mathbb{R}^3)$$

which is to say:

$$\mathrm{d}^q_{\mathrm{CS}}(\mathcal{O}_L) = 0$$

Therefore it makes sense to inquire about it's expectation value. [24] Note that as any link in $\mathbb{R}^3$ is null-homotopic,

Lemma 6.

$$< \mathcal{O}_L >= 0$$

So that this theory has nothing to say about a *single* link. What about two?

That is, what about the "product" [25] of two observables given by disjointly embedded knots. [26] Naively, one would like to argue that because their expectation values are both zero, the expectation value

$$\begin{aligned} < \mathcal{O}_{L_1} \cdot \mathcal{O}_{L_2} > &=< \mathcal{O}_{L_1} >< \mathcal{O}_{L_2} > \\ &= 0 \cdot 0 = 0 \end{aligned}$$

This would be true if our observables formed an algebra. Fortunately it doesn't! Of course, when $\hbar = 0$, this is the case.

In the language of field theory, these observables may be *correlated.* [27] The computation may clarify why this is the case. In fact,

Lemma 7. *Given any two oriented, disjoint links* $L_1, L_2 : \mathbb{S}^1 \to \mathbb{R}^3$

$$< \mathcal{O}_{L_1} \mathcal{O}_{L_2} >_c = \hbar \cdot \mathrm{Link}(L_1, L_2)$$

Before we begin, let us recall one of the many equivalent classical constructions of this invariant. The primary ingredients are:

- Configuration spaces are functorial with respect to embeddings.
- The torus embeds into the configuration space of two points in a disjoint union of circles

Therefore, given any embedding: $L = L_1 \coprod L_2 : \mathbb{S}^1 \coprod \mathbb{S}^1 \to \mathbb{R}^3$, we obtain a map:

$$\begin{aligned} \Gamma_{1,2} : \mathbb{T}^2 \hookrightarrow \mathrm{Conf}_2(\mathbb{S}^1 \coprod \mathbb{S}^1) &\xrightarrow{\mathrm{Conf}_2(L)} \mathrm{Conf}_2(\mathbb{R}^3) \mathbb{S}^2 \\ (\theta_1, \theta_2) &\longmapsto \frac{L_1(\theta_1) - L_2(\theta_2)}{|L_1(\theta_1) - L_2(\theta_2)|} \end{aligned}$$

Using this map, we can integrate (over the torus) the pullback of a normalized volume form on S^3. We define this number to be the linking number of these embedding links.

In a more algebro-topological language, [28] the linking number is the degree of the above map, and is a whole number. Therefore, as this definition is continuous, this number is isotopy invariant. A theorem of Gauss asserts that this quantity may be computed as a signed counted of intersection of a disk bounding one of the links with the other. Therefore, the linking number of links which are not linked is zero.

We now give two proofs of the above theorem.

Proof. Recall that for any Chern Simons field A:

$$\mathcal{O}_{L_1}(A) = \int \eta_{L_1} \wedge A$$

Of course, this value is zero! We can see this by constructing a nullhomotopy, which can be conveniently modelled by choosing a disk, $D_1 : \mathbb{D}^2 \to \mathbb{R}^3$, bounding L_1. We choose [29] D_1 so that it intersects L_2 transversely.

Recall that our goal is equivalent to finding a solution to:

$$\mathrm{d}^q(?) = \mathcal{O}_{L_1} \cdot \mathcal{O}_{L_2} - < \mathcal{O}_{L_1} \cdot \mathcal{O}_{L_2} >$$

Where the term on the right is some polynomial in $\hbar$. We do so by computing the differential of the obvious choice:

$$\begin{aligned}
\mathrm{d}^q(\mathcal{O}_{D_1} \cdot \mathcal{O}_{L_2}) &\simeq \mathcal{O}_{L_1} \cdot \mathcal{O}_{L_2} - \hbar \Delta_{\mathrm{BV}}(\mathcal{O}_{D_1} \cdot \mathcal{O}_{L_2}) \\
&\simeq \mathcal{O}_{L_1} \cdot \mathcal{O}_{L_2} - \hbar \{\mathcal{O}_{D_1}, \mathcal{O}_{L_2}\} \\
&\simeq \mathcal{O}_{L_1} \cdot \mathcal{O}_{L_2} - \hbar \sum_{D_1 \cap L_2} \pm 1 \\
&\simeq \mathcal{O}_{L_1} \cdot \mathcal{O}_{L_2} - \hbar \mathrm{Lk}(L_1, L_2)
\end{aligned}$$

where the signs on the second line are determined by the right hand rule. The first line follows from the fact that d_{dR} is a formal first order operator. The second line follows from our geometric description the the Poincare pairing.

Recall that we can also express $\mathcal{O}_{L_i}$ as the integral of a two form, η_i, concentrated within a small tubular neighborhood of L_i, integrating to 1 along the normal directions. Therefore, given a choice of a right-inverse of the deRham differential, an identical analysis shows that:

$$\mathcal{O}_{L_1} \cdot \mathcal{O}_{L_2} \simeq \hbar \int_{\mathbb{R}^3} \eta_1 \mathrm{d}_{\mathrm{dR}}^{-1}(\eta_2)$$

A convenient choice for $\mathrm{d}_{\mathrm{dR}}^{-1}$ is provided by convolution with the pullback of the standard normalized volume form along Gauss map $\mathrm{Conf}_2(\mathbb{R}^3) \simeq \mathbb{R} \times \mathrm{S}^2 \to \mathrm{S}^2$. The formula for this in standard coordinates is:

$$\omega(x) = \frac{1}{8\pi} \varepsilon_{ijk} \frac{x^i dx^j dx^k}{|x|^3}$$

Localizing this integral around tubular neighborhoods, and integrating along the normal directions of the links show that this is computed as the integral written down by Gauss. Note that as $\mathbb{R}[\hbar]$ is concentrated in degree zero, the two numbers which we computed must be equal. This is the precisely Gauss' classical expression of the linking number as an integral. □

Note that the correlation function contains only $\hbar$ dependent terms. Therefore, the linking number is not seen by classical observables. Here, we see that our quantization procedure allowed us to detect this geometric information, in a manifestly invariant manner.

7. Relation to the Functional Integral

In the continuum limit one has infinitely many integration variables... Problems with infinitely may variables can be very difficult to solve.

—Kenneth Wilson, [W]

Now that the math is over, we'd like to say a few words about how our discussion relates to the quantization procedure provided by Feynman's functional integral, and associated diagrammatics.

The raison d'etre of the functional integral is the expression of the theory's partition function and effective action/Hamiltonian. These are function on the dual space of the fields (sources), and serve as the generating function for expectation values of all observable quantities of the theory. More explicitly, given a local classical action, S_{cl}, the parition function is defined to be:

$$Z[\mathcal{J}] = \int \mathcal{D}A \cdot \mathrm{e}^{-\mathrm{S}_{\mathrm{cl}}(A) + <\mathcal{J}, A>}$$

while the effective action is:

$$\mathrm{S}^{\mathrm{eff}} = \log Z[\mathcal{J}]$$

where the integral is over all physically distinguisble degrees of freedom, and $\mathcal{J}$ is a linear function of our fields. We emphasize that we are not integrating over $\mathcal{J}$; in Feynman's words, $\mathcal{J}$ is "classical".

The fundamental theorem of perturbative field theory says that upon decomposing the classical actioon into quadratic (free) and higher order (interacting) pieces:

$$S_{\text{cl}} = \frac{1}{2} < -, Q- > + S_{\text{int}}$$

the partition function may be Taylor expanded [30] and expressed as:

$$Z[\mathcal{J}] = \mathrm{e}^{-S_{\text{int}}(\frac{\delta}{\delta \mathcal{J}})} \int \mathcal{D}A \cdot \mathrm{e}^{-\frac{1}{2}<A,QA>+<\mathcal{J},A>}$$

Feynman diagrams organize the computation of the Taylor coefficients of this function. [31] In the standard popular science interpretation, these diagrams organize a sum of all histories, with particles interacting/scattering/colliding/decaying in a manner specified by the Taylor coefficients of the interaction terms. Following this logic, this theory explains the origin of nonlocal interactions: they occur due a coupling to the same diffuse field. Their interaction with this field creates an exchange of particles, thus producing a force. For a more detailed exposition on this interpretation, the reader is invited to consult [Z]. [32]

We emphasize that, although these diagrams are a very useful organizational tool, their specific form has no physical meaning. No experiment will verify the contribution of any single diagram for a generic theory. For example, in many descriptions, Q does not act invertibly on the initial degrees of freedom, thus requiring one to add additional (redundant) degrees of freedom in order to compute the integral. One must then check that these additional degrees of freedom do not alter the *physical* predictions. Therefore, making sense of these expressions requires hard, subtle effort. [33]

However, no matter how one defines it, the partition function should satisfy various functional equations. For example, the translation invariance of $\mathcal{D}A$ should give the Schwinger-Dyson equation:

$$< \frac{\delta \mathcal{O}}{\delta x} > = < \mathcal{O} \cdot \frac{\delta S_{\text{cl}}}{\delta x} >$$

which may be used to relate the partition function to a Green's function for Q:

$$Z[\mathcal{J}] = e^{-\frac{\delta}{\delta \mathcal{J}} S_{\text{int}}} \cdot e^{\frac{1}{2}<\mathcal{J},Q^{-1}\mathcal{J}>}$$

Therefore, one does not need to make sense of that contribution to the partition function, as we are free to compute a Green's function through other means.

More generally, the divergence operator for $\mu = \mathcal{D}A \cdot e^{-\mathrm{S}_{\text{cl}}(A)}$ generates equations of the form:

$$< \mathrm{Div}_{\mu}(X) > = 0$$

for any vector field X on the space of fields. This follows from the fundamental theorem of calculus.

The basic idea of our mathematical model is that it's easier to express these functional equations than it is to define the functional integral. Moreover, we can construct natural algebraic structures which express and manipulate these functional equations. One can then hope to write down enough independent equations to compute the desired quantities. Feynman diagrammatic calculations emerge in this picture through attempting to understand how various compressed representations of these structures are related to each other [1]. [34] In our case, we used a Lie/Commutative structure.

Taking the ansatz that our differential corresponds to the diverenge operator of some measure, we can exploit the relationship between the divergence operator and the classical action to reverse engineer the classical Chern-Simons action:

$$\mathrm{S}_{\text{CS}}(A) = \frac{1}{2} < A, \mathrm{d}_{\text{dR}} A >$$

Therefore, the above formula gives that the linking number may be obtained via a functional integral:

$$\mathrm{Lk}(L_1, L_2) = \frac{\delta^2}{\delta\mathcal{O}_{L_1}\delta\mathcal{O}_{L_2}}|_{J=0}\mathrm{S}_{\mathrm{CS}}^{\mathrm{eff}} =< (\mathcal{O}_{L_1} - < \mathcal{O}_{L_1} >)(\mathcal{O}_{L_2} - < \mathcal{O}_{L_1} >) >$$

The reader interested in the derivation of linking number through functional integrals is invited to consult [BK].

8. Final Remark

As a final remark, we to make the following assertion:

Linking numbers are not a special feature of Chern-Simons Theory. They are a *generic* feature of factorization homology.

This is exemplified (for those suitably initiated) in the following, which was given as an exercise at the 2014 West Coast Algebraic Topology Summer School:

Let $\mathrm{S}^1 \xrightarrow{e} \mathbb{R}^3 \xleftarrow{e'} \mathrm{S}^1$ be two disjoint knots. Let A be an E_3-aglebra in $(\mathrm{Ch}_{\mathbb{Q}}, \otimes)$, the symmetric monoidal category of chain complexes over the rational with tensor product. Use the defining properies of factorization homology to construct the sequence of maps:

$$\mathrm{C}_*(\mathrm{S}^1 \times \mathrm{S}^1; \mathbb{Q}) \otimes A \otimes A \to \int_{\mathrm{S}^1 \times \mathrm{S}^1} A \to \int_{\mathbb{R}^3} A \simeq A$$

Show that this composite map factors as the map

$$(A \otimes A)[2] \xrightarrow{Lk(e,e')} (A \otimes A)[2] \xrightarrow{[-,-]} A$$

which is the linking number e and e' times the Lie bracket operation on A.

We invite the reader to consult [AF] for a rigorous introduction to factorization homology.

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[W] Wilson, Kenneth. The Renormalization Group: Critical Phenomena and the Kondo Problem.
[Z] Zee, Anthony. Quantum Field Theory in a Nutshell.

Notes

1. See [M] for a more general introduction to this theory, and it's relation to more standard approaches to field theory. Essentially, the Batalin-Vilkovisky formalism is a symplectic approach to field theory, amenable to homotopical methods.

2. realized experimentally by engineering near impossibly low temperature

3. As opposed to, for example, a spin system description. See [S] for a wonderful introduction to such approaches.

4. This really needs to be treated as a single object. For example, the ellipticity of the deRham differential (which maps *between* the various graded pieces) permits the exploitation of elliptic regularity and heat kernel methods, thus avoiding irrelevant divergent quantities arising from our choice of model. In the discussion below, everything should be taken as "up to higher coherent homotopy". In practice, this means up to equal up to smoothing operators vanishing on certain subspaces. Despite being infinite-dimensional Frechet spaces, these objects may be placed in a convenient setting in which standard homological arguments are valid, and the standard spectral sequences behave as expected (notably those arising from weight filtrations).

5. From this point forward, we will omit explicit reference to the deRham differential on forms. We invite the reader to check that our maps will be suitably compatible with the differentials. Thinking of the deRham differential as a linear differential operator $((\mathrm{d}x^i) \cdot \partial_i)$ generating some sort of flow, we may say that all our maps will be suitably equivariant.

6. Forms with disjoint support are orthogonal with respect to this pairing. In this sense, this pairing is "local".

7. It's an enlightening exercise to go through the classical theorems necessary to prove this.

8. A delicate engineering of semantics permits one to adopt the perspective that Poincare pairing of differential forms as a symmetric, quadratic function on our protagonist, taking values in numbers concentrated in a non-connective degree corresponding to the top dimension of the manifold of interest. In other words, "shifted".

9. We may evocatively write:

   $$\Omega_c^\bullet[1] \simeq \Omega_c^\bullet \otimes u(1)[1]$$

   As compactly supported forms admit a nonunital commutative multiplication, we can view $\Omega_c^\bullet \otimes u(1)$ as a Lie algebra. [35]

10. by which we mean chain complex + appropriate adjectives

11. The author could not locate a classical source in symplectic geometry which explicitly state that the symplectic forms to take values in degree 0. Therefore, we will omit the term shifted, as the relevant degree will always be implicitly determined in our discussion.

12. Our fields come with a spin 1 representation, and therefore may be called photons. Of course, the dynamics will be quite different from Maxwell's electromagnetism. This system behaves more "exotically".

13. The symmetries of our system are encoded in the definition of a cosheaf on the site of n-manifolds.

14. Recall that Lie algebra cohomology of an abelian/trivial Lie algebra is a free commutative algebra, generated by the dual of a shift of the Lie algebra. In the lingua franca of field theory, our theory is "free". This is mirrored by the general fact that the commutative algebra of functions on a linear object (e.g solutions to linear "equations of motion") is free, with an abelian Lie structure on it's shifted tangent space.

15. We invite the reader to think of this as object as a polynomial of currents.

16. We can imagine such an object as follows. V describes a family of $\dim V$ types of particles. The symmetrization of our monomial describes k external, identical particles localized at the unordered configuration $(x_0, \ldots, x_k)_{\Sigma_k}$.

17. In some sense, in this setting the duality between the product on cohomology and transverse intersection of submanifolds is the duality between a symplectic form and it's Poisson bracket.

18. One way to formally obtain this as follows. Multiplying our Poisson bracket by a formal parameter $\hbar$ gives a map:

    $$\mathrm{Sym}^2\Big((\Omega_c^\bullet[1])^\vee[1]\Big) \xrightarrow{\{\cdot,\cdot\}} (\mathbb{R}\cdot\hbar)[n]$$

    which extends to a map of cocommutative coalgebras:

    $$\mathrm{CE}_*((\Omega_c^\bullet[1])^\vee) \to \mathrm{CE}_*(\mathbb{R}\cdot\hbar[n-1])$$

    where we are thinking of both sides of the map as cofree cocommutative coalgebras. The Heisenberg Lie algebra is then defined as the (-1)-shifted tangent space of the pullback of the above map.

19. This is formally analogous to second quantization, which forms a Fock space of identicle particles from the space of one-particle states.

20. This is formally analogous to second quantization, which forms a Fock space from the Hilbert space of one-particle states

21. They were constructed via a bar construction, which inherits a cardinality filtration.

22. Vacuum

23. which may not always exist, in agreement with classical probability theory. Of course, we can always define the expectation value of such an observable for which there does not exist a lift to be zero.

24. The fact that labelled curves give rise to observables is one indication that we are discussing a gauge theory.

25. The language of ∞-operads allows us give an algebraic structure in which this is the product. In other words, remove the scare quotes. See [G][CG] for a detailed discussion along these lines.

26. One can imagine these links as arising from the difference of two distinct trajectories of two particles. Of course, in order to compare these trajectories, they must start and end at the same location. Hopefully, this gives some hint of how these quantities are related to the exchange "statistics" via an Aharanov-Bohm phase. See [D] for a more responsible discussion.

27. In the sense of probability theory.

28. For an approach to linking number via homotopy theory, the reader is invited to consult [GM]. The reader may consult [Ma] for an approach via Massey products.

29. Because we can. [36]

30. so that the formula below is only valid when the interacting term is small

31. In Feynman's languare, this is the "influence functional". The fact that we are performing this integral over the A variables implicitly indicates that we are restricting ourselves to observations of $\mathcal{J}$, viewing A as an external force. A is the "environment", so that the partition function encodes the influence of the environment on $\mathcal{J}$. Therefore, this procedure compresses our model, resulting in the loss of information. [37]

32. We note that the same type of computations may be made when we "know" that elementary particles are not created, namely the study of collective phenomena . It's conventional to refer to such particles as "quasi-particles". For example, in this field of discourse, producing a sound by striking a table produces "phonons" which are exchanged from the hand to the eardrum.

33. This may dramatically alter one's conceptualization of the behavior of the system, introducing "ghosts" into the theory. The zero forms in our description were one such example of this. Such cruel language should not be applied to such beautiful objects in order to preserve a naive ontological dogma.

34. In other words, Feynman diagrams compute homotopy transfers of algebraic structures.