ALEXANDER DUALITY IN IMAGE ANALYSIS

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1. Introduction:

The goal of these notes are to efficiently compute the first homology of the superlevel sets of large, high resolution grey-scale images. The primary tools are the locality of cohomology and Poincare Duality.

The first step consists of application of Alexander duality to naturally relate the first homology of the dark regions with the connected components of the light region.

The second step involves the parallelization of the computation of the connected components of the light region. In other words, splitting the image into overlapping patches, (independently) computing the connected components of the patches and their overlaps, and gluing together these results.

In this context, the gluing operation incarnates itself as computing the kernel of a matrix. The matrix encodes how connected regions share common overlap(s). The entries of this matrix do not correspond to pixels, but to connected light regions. As this matrix is built from geometric data, and it is sparse.

These identifications are natural with respect to the related choices of the hyperparameter corresponding to a chosen notion of light/dark, and therefore carry over to the computation of persistent diagrams.

2. Alexander Duality

A (nice) closed subspace X of S^n, deconstructs S^n as a union of X and the closure of it's complement in S^n, X^c, glued together along their common boundary:

$$S^n \simeq X \coprod_{\partial X = \partial X^c} X^c$$

These notes represent an application of:

Theorem 1. *Alexander Duality:*
There exists a natural equivalence:

$$\tilde{\mathrm{H}}_k(X) \simeq \tilde{\mathrm{H}}^{n-k-1}(X^c)$$

This follows from a simple geometric identification, recognizing that the inclusion $X \hookrightarrow \mathrm{S}^n$ induces an equivalence:

$$X/\partial X \simeq \mathrm{S}^n/X^c$$

along combination of two fundamental properties of homology: Poincare Duality and the locality of (co)homology

In the discussion below, I'll be playing fast an loose about whether I'm talking about a space, it's interior, or it's closure.

2.1. **Case of Interest.** We are primarily interested in the case when $n = 2$ and $k = 1$, in which case Alexander Duality gives an equivalence:

$$\mathbb{R}^{\pi_0(X)} \simeq \tilde{\mathrm{H}}^0(X^c) \simeq \tilde{\mathrm{H}}_1(X)$$

We view this as useful as useful formula to compute the right hand side in terms of the left hand side, which may be computed through more computationally efficient (pre-existing) methods.

One way to model a greyscale image is as a compactly supported function:

$$S : \mathbb{R}^2 \to [0, 1]$$

which we'll refer to as the saturation. Here, compactly supported means that it takes a fixed value outside of some bounded region, which we'll take to be zero. In other words, the saturation function extends continuously to the one point compactification of R^2, i.e. S^2:

$$S : \mathrm{S}^2 \to [0, 1]$$

A saturation scale s determines an overlapping decomposition of the space of possible saturation values:

$$[0, 1] \simeq [0, s] \coprod_s [s, 1]$$

Which in turn decomposes the two sphere into closed subsets:

$$\mathrm{S}^2 \simeq S^{-1}([0, s]) \coprod_{S^{-1}(\{s\})} S^{-1}([s, 1])$$

We'll refer to $S^{-1}([0, s]) =: X_s$ as the light region, and $S^{-1}([s, 1]) =: X_s^c$ as the dark region. Therefore, Alexander duality establishes a relation between first homology of the dark region with the number of connected components of the light region. In other words, a first homology class in the dark region corresponds to a "hole" in the light region. As one may imagine, the latter computation is significantly more accessible than the first.

In order for this equivalence to be useful in applications, this equivalence must vary functorially as we change or notion of dark vs. light. More precisely, as we decrease $s \in [0, 1]$.

Note that decreasing the saturation scale from s_0 to s_1 increases the size of the dark region, and decreases the size of the light region. In other words, there exists natural inclusions of the dark regions

$$X_{s_0} \hookrightarrow X_{s_1}$$

along with inclusions of the light regions *in the other direction*:

$$X_{s_0}^c \hookleftarrow X_{s_1}^c$$

This reflects an obvious fact: as the dark regions get larger, the light regions get smaller.

These induce maps relating the first homology of the dark regions:

$$\tilde{\mathrm{H}}_1(X_{s_0}) \to \tilde{\mathrm{H}}_1(X_{s_1})$$

and zeroeth cohomology of the light regions:

$$\tilde{\mathrm{H}}^0(X^c_{s_0}) \to \tilde{\mathrm{H}}^0(X^c_{s_1})$$

Therefore, establishing the naturality of Alexander duality gives an equivalence of the relevant persistence diagrams. Here, the persistence parameter is given by the saturation scale. Proving naturality of Alexander duality is not particularly difficult, as it's built from natural constructions: Poincare Duality and the coboundary map in a natural long exact sequence in cohomology.

Theorem 2. *Alexander Duality yields an equivalence of persistence diagrams:*

$$\tilde{\mathrm{H}}_1(X_s) \simeq \tilde{\mathrm{H}}^0(X^c_s)$$

Where we are viewing both sides as functors:

$$[0,1] \to \mathrm{Vect}$$

2.2. **An Intuitive Explanation of the Equivalence.** For example, we imagine $\eta \in \tilde{\mathrm{H}}^0(X^c_s)$ as a function which is constant on a connected component of the light region X^c, and zero elsewhere. This is the $\tilde{\mathrm{H}}^{2-1-1}(X^C)$ side of Alexander duality. We can then extend this function to the entire image S^2. We then compute gradient of this extension. As the function was constant in the light region, the gradient of it's extension to the entire region is contain within the dark region.

We take the vector field perpendicular to this gradient, using the right hand rule. In other words if we point our right index finger in the direction of this gradient, our new vector field points in the direction of our thumb. The homology class is a loop which integrates to 1 along this vector field. Note it suffices to find a loop which integrates to a nonzero value.

The first paragraph is the Mayer-Vietoris principle, while the second is Poincare Duality.

2.3. **An Example: Jordan Curve Theorem.** The Jordan Curve theorem states that an embedded curve inside of $\mathbb{R}^2$ the plane into two disjoint pieces. We now outline how this theorem is an application of the Alexander Duality.

First, note that the embedded:

$$\gamma : \mathbb{R} \to \mathbb{R}^2$$

extends to their one-point compactifications:

$$\mathrm{S}^1 \simeq \mathbb{R}^+ \xrightarrow{\gamma^+} (\mathbb{R}^2)^+ \simeq \mathrm{S}^2$$

Our assumption that γ was an embedding shows that it's image is equivalent to S^1. Taking X to be a closed neighborhood of the image of γ shows that:

$$\tilde{\mathrm{H}}^0(X^c) \simeq \tilde{\mathrm{H}}_1(X) \simeq \tilde{\mathrm{H}}_1(\mathrm{S}^1)$$

The number of connected compents of a space Y is $1+\dim(\tilde{\mathrm{H}}^0(S))$, and the dimension of $\tilde{\mathrm{H}}_1(\mathrm{S}^1)=1$ implies that the number of connected components of X^c is two. As number of connected components of $\mathbb{R}^2-\mathrm{Im}(\gamma)$ coincides with that of X^c allows us to conclude the desired theorem.

2.4. **The Equivalence.** Note that one only needs an explicit description of the equivalence if one either desires to understand why this is true, or translate between connected components of X^c and loops of X.

Poincare Duality gives an equivalence, for any oriented manifold M:

$$\tilde{\mathrm{H}}_k(M)\simeq\tilde{\mathrm{H}}_c^{n-k}(M)\simeq\tilde{\mathrm{H}}^{n-k}(M^+)$$

Where M^+ denotes the one-point compactification of M.

The one-point compactification is relevant to our examples as:

$$\mathrm{int}(X)^+\simeq X/\partial X$$

Which implies that:

$$\begin{aligned}\tilde{\mathrm{H}}_1(X)&\simeq\tilde{\mathrm{H}}^1(X/\partial X)\\&\simeq\tilde{\mathrm{H}}^1(\mathrm{S}^2/X^c)\end{aligned}$$

Conceptually, the first equivalence is given by taking a cycle in X, and finding (the dual of) a chain perpendicular, which starts and ends on ∂X. The second equivalence is due to the equivalence:

$$X/\partial X\simeq\mathrm{S}^n/X^c$$

stated in the first section.

Therefore, we need to relate the cohomology of S^2/X^c to the cohomology of X^c. We do so by relating the cohomology of these two spaces with the cohomology of S^2, which couldn't be simpler. Formally, this relationship is given by noting the following pushout diagram:

$$\begin{array}{ccc}X^c & \longrightarrow & *\\ \big\downarrow & & \big\downarrow\\ \mathrm{S}^2 & \longrightarrow & \mathrm{S}^2/X^c\end{array}$$

Standard techniques in algebraic topology shows that the boundary map in the long exact sequence associated to the deconstruction above:

$$\tilde{\mathrm{H}}^{*+1}(\mathrm{S}^2/X^c)\simeq\tilde{\mathrm{H}}^*(X^c)$$

is an equivalence when $*<2$.

This isomorphism admits an explicit geometric description. Given an closed 0-cochain on X^c, η, this map is produced as follows:

- Find an extension, $\tilde{\eta}$, of η to all of S^2.
- Compute the coboundary of this extension $d\tilde{\eta}\in\mathrm{C}^1(\mathrm{S}^2)$
- Find a 1-cochain γ on S^2/X^c whose restriction to S^2 is $d\tilde{\eta}$

Less formally, the isomorphism is given by taking coboundary operator.

3. A Local-to-Global Computation of The Connected Components

Recall the Euler characteristic of a space[1]:

$$\chi(X) = \dim(\mathrm{H}^k(X))(-1)^k$$

This satisfies the following inclusion-exclusion principle. Given a decomposition of a space into a union of subspaces U and V:

$$\chi(X) = \chi(U) + \chi(V) - \chi(U \cap V)$$

Which may be viewed as an analogue of the classic formula in information theory:

$$H(X, Y) = H(X) + H(Y) - I(X, Y)$$

Note that this formula is not true upon replacing $\chi(-)$ with $\dim(\mathrm{H}^0(-))$.

However, there does exist an analogous relationship, as long as one takes into account the algebraic and categorical structure of $\mathrm{H}^*(-)$. One articulation of this is the "Mayer-Vietoris Principle", which expresses the "locality" of cohomology:

Theorem 3. *Mayer-Vietoris*

The following is a pullback diagram:

$$\begin{array}{ccc} \mathrm{C}^*(X) & \longrightarrow & \mathrm{C}^*(U) \\ \downarrow & & \downarrow \iota_U^* \\ \mathrm{C}^*(V) & \xrightarrow{\iota_V^*} & \mathrm{C}^*(U \cap V) \end{array}$$

Equivalently:

$$C^*(X) \simeq \ker\Big(\mathrm{C}^*(U) \oplus \mathrm{C}^*(V) \overset{\iota_U^* - \iota_V}{\to} \mathrm{C}^*(U \cap V)\Big)$$

This has the following consequence:

$$\mathrm{H}^0(X) \simeq \ker\Big(\mathrm{H}^0(U) \oplus \mathrm{H}^0(V) \overset{\iota_U^* - \iota_V}{\to} \mathrm{H}^0(U \cap V)\Big)$$

In English, this states that every locally constant function of X is uniquely determined by a locally constant on U and a locally constant on V, which agree when restricted to $U \cap V$. This is sometimes expressed as a local-to-global method of computation.

Here, we see two interrelated algebraic and categorical structures at place, first excated by Noether. First, one must view H^0 as more than it's dimension (a number), but a vector space, so that we can write down the maps $\iota^*_{(-)}$. Second, we must think of H^0 categorically, (i.e. as a functor), as we need the inclusions to give rise to the linear maps $\iota^*(-)$, and be able to express $H^0(X)$ as a kernel, which the author views as a categorical construction.

In our case of interest, this means that we can parallelize the compution of $\mathrm{H}^0(X^c)$, by breaking down the image into patches, and applying the formula inductively. Combining this with the Alexander duality, we have produced a parallelization of the computation of $\mathrm{H}^1(X)$.

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[1]Here, we are using Einstein notation, i.e. an implicit sum over indices